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A shape-preserving quasi-interpolation operator satisfying quadratic polynomial reproduction property to scattered data

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ABSTRACT

In this paper, we construct a univariate quasi-interpolation operator to non-uniformly distributed data by cubic multiquadric functions. This operator is practical, as it does not require derivatives of the being approximated function at endpoints. Furthermore, it possesses univariate quadratic polynomial reproduction property, strict convexity-preserving and shape-preserving of order 3 properties, and a higher convergence rate. Finally, some numerical experiments are shown to compare the approximation capacity of our quasi-interpolation operator with that of Wu and Schaback's quasi-interpolation scheme.

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1. Introduction

With the development of radial basis functions, more and more researchers construct interpolation functions with them, and have obtained better results. For example, [2,9,10] introduced some interpolation functions by using radial basis functions. In particular, the multiquadric function of first degree was proposed in [8], which performed well in many fields, and Franke [5] also showed that the interpolation functions in virtue of multiquadric functions were the best in accuracy and efficiency by performing many numerical experiments. However, when the number of interpolation points is very large, the interpolation matrix might be ill-conditioning. Compared with interpolation, the quasi-interpolation method not only can avoid the ill-conditioning problem, but also possesses the polynomial reproduction property and better shape-preserving properties. On the basis of these advantages of quasi-interpolation method, researching how to construct a quasi-interpolation operator with better properties to non-uniformly distributed data has been a hot topic recently. Because constructing multivariate quasi-interpolation operators is much more difficult, the authors mostly study the univariate quasi-interpolation operators.

For given a function $f(x) : C[a, b] \to \mathbb{R}$, on scattered data $\{x_j\}_{j=0}^n$ and points $\{(x_j, f(x_j))\}_{j=0}^n$, where

$$\{x_j | a = x_0 < x_1 < \cdots < x_n = b\}, \qquad h = \max_{0 \le j \le n-1} (x_{j+1} - x_j), \tag{1}$$

the quasi-interpolation operator of $f(x)$ has the general form:

$$\mathcal{L}(x) = \sum_{j=0}^{n} f(x_j)\varphi_j(x), \quad x \in [a, b],$$

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where $\varphi_j(x)$ are some given basis functions. And there are many famous papers, which have constructed some univariate quasi-interpolation operators with radial basis functions. For instance, [7] constructed a quasi-interpolation operator $\mathcal{L}_1(x)$ by shifts of multiquadric function of first degree, and this operator satisfied the linear polynomial reproduction property and good approximation properties, and [11] showed $\mathcal{L}_1(x)$ possessed convexity preserving properties. However, $\mathcal{L}_1(x)$ requires the derivatives of $f(x)$ at endpoints, it is not very convenient for practical purposes. Wu and Schaback [11] constructed a simple quasi-interpolation operator $\mathcal{L}_2(x)$ by shifts of multiquadric function of first degree, but its approximation order was 2 at most. Based on [11], Leevan Ling [3] proposed a multilevel univariate quasi-interpolation scheme using multiquadric basis functions, and proved the proposed scheme converged with a rate of $O(h^{2.5}\log\ h)$ as $c = O(h)$. [12] constructed four types of quasi-shape-preserving quasi-interpolation operators by shifts of cubic MQ-B-Splines, and discussed their convergence capacities, but they couldn't satisfy the polynomial reproduction property. [1] gave a conclusion that the approximation order of a quasi-interpolation scheme had close relations with the polynomial reproduction property. By studying those above-mentioned quasi-interpolation operators, we find that the operators in [3,7,11] only satisfy the linear polynomial reproduction property, and the operators in [7] and [12] require the derivatives of $f(x)$ at endpoints, it is not very convenient for practical operations. Therefore, the aim of our paper is to construct a much simpler shape-preserving quasi-interpolation operator by shifts of cubic multiquadric functions, and this quasi-interpolation operator can also satisfy quadric polynomial reproduction property and better approximation capacity, so that it is convenient for people in various applications.

The organization of the rest of this paper is as follow: In Section 2, we introduce some necessary definitions and known conclusions. In Section 3, we first construct a quasi-interpolation operator by shifts of cubic multiquadric function to non-uniformly distributed data, then discuss the polynomial reproduction property, shape-preserving properties and the approximation capacity in detail. In Section 4, we give some numerical experiments to compare the approximation capacity of our quasi-interpolation scheme with that of Wu and Schaback's quasi-interpolation scheme, and also verify the convergence rate of our quasi-interpolation operator by examples. In the last section, we give a conclusion and the following work of the authors.

2. Preliminaries

This section contains some necessary definitions and known conclusions. We quote the definition of multiquadric (MQ) function in [6].

Definition 1. Consider points sequence (1), let $k \in \mathbb{N}$, $c > 0$, multiquadric functions of degree $2k-1$ ($2k$ order) is defined by

$$\phi(x, 2k) = (x^2 + c^2)^{(2k-1)/2},$$

where c is called shape parameter, and

$$\phi_{j,2k}(x) = \phi(x - x_j, 2k)$$

is a shift of $\phi(x, 2k)$ centered at x_j.

Definition 2. Suppose $\mathcal{L}(x)$ is an approximation of the function $f(x)$ and constructed by data points $\{(x_j, f(x_j))\}_{j=0}^n$, if the divided difference of degree k of data points $\{(x_j, f(x_j))\}$ is nonnegative (non-positive), and the derivative of degree k of $\mathcal{L}(x)$ is also nonnegative (non-positive), then $\mathcal{L}(x)$ is called to be shape-preserving of order k. Especially, shape-preserving of first degree is just monotonicity-preserving, and shape-preserving of second degree is just convexity-preserving.

In the following, we introduce some known quasi-interpolation operators.

For given scattered data $a = x_0 < x_1 < \cdots < x_n = b$, Wu and Schaback [11] constructed a quasi-interpolation operator $\mathcal{L}_2(x)$ by the shifts of multiquadric function of first degree $\phi_j(x) = \sqrt{(x-x_j)^2 + c^2}$,

$$\mathcal{L}_2(x) = f(x_0)\alpha_0(x) + f(x_1)\alpha_1(x) + \sum_{j=2}^{n-2} f(x_j)\varphi_j(x) + f(x_{n-1})\alpha_{n-1}(x) + f(x_n)\alpha_n(x), \tag{2}$$

where

$$\alpha_0(x) = \frac{1}{2} + \frac{\phi_1(x) - (x - x_0)}{2(x_1 - x_0)},$$
$$\alpha_1(x) = \frac{\phi_2(x) - \phi_1(x)}{2(x_2 - x_1)} - \frac{\phi_1(x) - (x - x_0)}{2(x_1 - x_0)},$$
$$\varphi_j(x) = \frac{\phi_{j+1}(x) - \phi_j(x)}{2(x_{j+1} - x_j)} - \frac{\phi_j(x) - \phi_{j-1}(x)}{2(x_j - x_{j-1})},$$
$$\alpha_{n-1}(x) = \frac{(x_n - x) - \phi_{n-1}(x)}{2(x_n - x_{n-1})} - \frac{\phi_{n-1}(x) - \phi_{n-2}(x)}{2(x_{n-1} - x_{n-2})},$$

$$\alpha_n(x) = \frac{1}{2} + \frac{\phi_{n-1}(x) - (x_n - x)}{2(x_n - x_{n-1})}.$$

And they proved $\mathcal{L}_2(x)$ possessed linear polynomial reproduction property, strict monotonicity-preserving and convexity-preserving properties, and the convergence order of $\mathcal{L}_2(x)$ was 2 at most. [12] firstly extended the points sequence (1) to get a new sequence

$$x_{-3} = x_{-2} = x_{-1} = a, \qquad x_{n+1} = x_{n+2} = x_{n+3} = b,$$

then they used $\varepsilon_j = \sum_{i=j-1}^{j+1} \frac{x_i}{3}$ to get a point sequence

$$a = \varepsilon_{-1} < \varepsilon_0 < \cdots < \varepsilon_{n+1} = b. \tag{3}$$

Zhang and Wu [12] constructed a quasi-interpolation operator $\mathcal{L}_3(x)$ to the point sequence (3) by cubic multiquadric function $\phi(x) = (x^2 + c^2)^{\frac{3}{2}}$,

$$\begin{aligned} \mathcal{L}_3(x) = {} & \beta_{-1}(x) f(\varepsilon_{-1}) + \gamma_{-1}(x) f'(\varepsilon_{-1}) + \lambda_{-1} f''(\varepsilon_{-1}) + \sum_{j=0}^{n} f(\varepsilon_j)\varphi_j(x) \\ & + \beta_{n+1}(x) f(\varepsilon_{n+1}) + \gamma_{n+1}(x) f'(\varepsilon_{n+1}) + \lambda_{n+1} f''(\varepsilon_{n+1}) \end{aligned} \tag{4}$$

where

$$\begin{aligned} \varphi_j(x) &= \frac{x_{j+2} - x_{j-2}}{2}[x - x_{j+2}, x - x_{j+1}, x - x_j, x - x_{j-1}, x - x_{j-2}]\phi(x), \\ \beta_{-1}(x) &= \frac{1}{2} - \frac{1}{2}[x - x_1, x - x_0, x - x_{-1}, x - x_{-2}]\phi(x), \\ \beta_{n+1}(x) &= \frac{1}{2} + \frac{1}{2}[x - x_{n+2}, x - x_{n+1}, x - x_n, x - x_{n-1}]\phi(x), \\ \gamma_{-1}(x) &= \frac{x - x_0}{2} - \frac{1}{12}\phi_0''(x), \qquad \gamma_{n+1}(x) = -\frac{x_n - x}{2} + \frac{1}{12}\phi_n''(x), \\ \lambda_{-1}(x) &= \frac{(x - x_0)^2}{4} - \frac{\phi_0'(x)}{12} + O(c^2) + O(h^2), \\ \lambda_{n+1}(x) &= \frac{(x_n - x)^2}{4} + \frac{\phi_n'(x)}{12} + O(c^2) + O(h^2), \end{aligned}$$

and $[x - x_{j+2}, x - x_{j+1}, x - x_j, x - x_{j-1}, x - x_{j-2}]\phi(x)$ is defined as Definition 2.1 in [12]. Furthermore, Zhang and Wu proved $\mathcal{L}_3(x)$ possessed the following properties.

Theorem 1. *Quasi-interpolation operator $\mathcal{L}_3(x)$ is quasi monotonicity-preserving and quasi convexity-preserving. Especially, if the points sequence (1) is uniformly distributed, $\mathcal{L}_3(x)$ is quasi shape-preserving of order up to* 3.

Theorem 2. *If the second order derivative of $f(x)$ is Lipschitz continuous, then the quasi-interpolation operator $\mathcal{L}_3(x)$ on $[x_0, x_n]$ satisfies*

$$\|f - \mathcal{L}_3\|_\infty \le \frac{\mu c^3}{12}[1 + 3c(x_n - x_0)] + O(c^2) + O(h^2),$$

where $\mu = \operatorname{ess\,sup}_{x_0 \le x \le x_n} |f'''(x)|$.

Obviously, the approximation order of $\mathcal{L}_3(x)$ is 2 at most, and the quasi-interpolation operator $\mathcal{L}_3(x)$ requires the derivatives of $f(x)$ at endpoints. It is not convenient for practical purposes. Therefore, we want to construct a simpler quasi-interpolation operator, which not only can reproduce quadric polynomials, but also possesses better shape-preserving property and higher approximation order in this paper.

3. Constructing a quasi-interpolation operator by shifts of cubic multiquadric function to scattered data

Suppose $f(x)$ is smooth enough, we will construct a quasi-interpolation operator $\mathcal{L}_d(x)$ by shifts of cubic multiquadric function $\phi_j(x) = [(x - x_j)^2 + c^2]^{\frac{3}{2}}$ to scattered points $\{x_j\}_{j=0}^n$ and data points $\{(x_j, f(x_j))\}_{0 \le j \le n}$, where

$$a = x_0 < x_1 < \cdots < x_n = b, \qquad h = \max_{0 \le j \le n-1}(x_{j+1} - x_j).$$

In particular, we take cubic multiquadric function $\phi(x) = \frac{1}{3}[x^2 + c^2]^{\frac{3}{2}}$ in this paper.

$$\begin{aligned}\mathcal{L}_d(x) &= f(x_0)\alpha_0(x) + f(x_1)\alpha_1(x) + f(x_2)\alpha_2(x) + \sum_{j=3}^{n-3} f(x_j)\alpha_j(x)\\ &\quad + f(x_{n-2})\alpha_{n-2}(x) + f(x_{n-1})\alpha_{n-1}(x) + f(x_n)\alpha_n(x)\end{aligned} \tag{5}$$

where

$$\begin{aligned}
\alpha_j(x) &= \left[\frac{\omega_j(x)-\omega_{j+1}(x)}{2(x_{j+2}-x_j)(x_{j+1}-x_j)} - \frac{\omega_{j-1}(x)-\omega_j(x)}{2(x_{j+1}-x_j)(x_{j+1}-x_{j-1})}\right]\\
&\quad - \left[\frac{\omega_{j-1}(x)-\omega_j(x)}{2(x_{j+1}-x_{j-1})(x_j-x_{j-1})} - \frac{\omega_{j-2}(x)-\omega_{j-1}(x)}{2(x_j-x_{j-1})(x_j-x_{j-2})}\right], \quad 3 \le j \le n-3,\\
\alpha_0(x) &= \frac{1}{2} + \frac{(x-x_0)^2-\omega_1(x)}{2(x_2-x_0)(x_1-x_0)} - \frac{x-x_0}{2(x_2-x_0)} - \frac{x-x_0}{2(x_1-x_0)},\\
\alpha_1(x) &= \left[\frac{\omega_1(x)-\omega_2(x)}{2(x_3-x_1)(x_2-x_1)} - \frac{(x-x_0)^2-(x_1-x_0)(x-x_0)-\omega_1(x)}{2(x_2-x_0)(x_2-x_1)}\right]\\
&\quad - \left[\frac{(x-x_0)^2-\omega_1(x)}{2(x_2-x_0)(x_1-x_0)} - \frac{x-x_0}{2(x_2-x_0)} - \frac{x-x_0}{2(x_1-x_0)}\right],\\
\alpha_2(x) &= \left[\frac{\omega_2(x)-\omega_3(x)}{2(x_4-x_2)(x_3-x_2)} - \frac{\omega_1(x)-\omega_2(x)}{2(x_3-x_2)(x_3-x_1)}\right]\\
&\quad - \left[\frac{\omega_1(x)-\omega_2(x)}{2(x_3-x_1)(x_2-x_1)} - \frac{(x-x_0)^2-(x_1-x_0)(x-x_0)-\omega_1(x)}{2(x_2-x_0)(x_2-x_1)}\right],\\
\alpha_{n-2}(x) &= \left[\frac{(x_n-x)^2+\omega_{n-2}(x)-(x_n-x_{n-1})(x_n-x)}{2(x_n-x_{n-2})(x_{n-1}-x_{n-2})} - \frac{\omega_{n-3}(x)-\omega_{n-2}(x)}{2(x_{n-1}-x_{n-2})(x_{n-1}-x_{n-3})}\right]\\
&\quad - \left[\frac{\omega_{n-3}(x)-\omega_{n-2}(x)}{2(x_{n-1}-x_{n-3})(x_{n-2}-x_{n-3})} - \frac{\omega_{n-4}(x)-\omega_{n-3}(x)}{2(x_{n-2}-x_{n-3})(x_{n-2}-x_{n-4})}\right],\\
\alpha_{n-1}(x) &= \left[\frac{x_n-x}{2(x_n-x_{n-1})} + \frac{x_n-x}{2(x_n-x_{n-2})} - \frac{(x_n-x)^2+\omega_{n-2}(x)}{2(x_n-x_{n-1})(x_n-x_{n-2})}\right]\\
&\quad - \left[\frac{(x_n-x)^2+\omega_{n-2}(x)-(x_n-x_{n-1})(x_n-x)}{2(x_n-x_{n-2})(x_{n-1}-x_{n-2})} - \frac{\omega_{n-3}(x)-\omega_{n-2}(x)}{2(x_{n-1}-x_{n-3})(x_{n-1}-x_{n-2})}\right],\\
\alpha_n(x) &= \frac{1}{2} + \frac{(x_n-x)^2+\omega_{n-2}(x)}{2(x_n-x_{n-1})(x_n-x_{n-2})} - \frac{x_n-x}{2(x_n-x_{n-1})} - \frac{x_n-x}{2(x_n-x_{n-2})},
\end{aligned}$$

and $\omega_j(x) = \frac{\phi_j(x)-\phi_{j+1}(x)}{x_{j+1}-x_j}$, $1 \le j \le n-2$.

By rewriting (5), we get another expression of $\mathcal{L}_d(x)$ as follows,

$$\begin{aligned}\mathcal{L}_d(x) &= \frac{1}{2}\sum_{j=1}^{n-2}(f[x_{j+2},x_{j+1},x_j] - f[x_{j+1},x_j,x_{j-1}])\omega_j(x) + \frac{1}{2}(f(x_0) + f[x_1,x_0](x-x_0) + f[x_2,x_1,x_0](x-x_0)^2)\\ &\quad + \frac{1}{2}(f(x_n) + f[x_n,x_{n-1}](x-x_n) + f[x_n,x_{n-1},x_{n-2}](x-x_n)^2)\\ &\quad - \frac{1}{2}f[x_2,x_1,x_0](x_1-x_0)(x-x_0) - \frac{1}{2}f[x_n,x_{n-1},x_{n-2}](x_n-x_{n-1})(x_n-x),\end{aligned} \tag{6}$$

where $f[x_{j+1}, x_j, x_{j-1}]$ denotes the divided difference of function $f(x)$.

3.1. The properties of quasi-interpolation operator $\mathcal{L}_d(x)$

In this section, we discuss the polynomial reproduction property and shape-preserving properties of the quasi-interpolation operator $\mathcal{L}_d(x)$ in detail.

Theorem 3. *Quasi-interpolation operator $\mathcal{L}_d(x)$ satisfies the quadric polynomial reproduction property, i.e.*

$$\sum_{j=0}^{n}(ax_j^2 + bx_j + c)\alpha_j(x) = ax^2 + bx + c, \quad a, b, c \in \mathbb{R}.$$

Proof. For any quadric polynomial $g(x) = ax^2 + bx + c$, we substitute $g(x_j)$ for $f(x_j)$ in (6), then $g[x_{j+2}, x_{j+1}, x_j] = g[x_{j+1}, x_j, x_{j-1}] = a$. Hence

$$\begin{aligned}\mathcal{L}_d(x) &= \frac{1}{2}\sum_{j=1}^{n-2}(g[x_{j+2}, x_{j+1}, x_j] - g[x_{j+1}, x_j, x_{j-1}])\omega_j(x)\\&\quad + \frac{1}{2}[ax_0^2 + bx_0 + c + a(x_1 + x_0)(x - x_0) + b(x - x_0) + a(x - x_0)^2]\\&\quad + \frac{1}{2}[ax_n^2 + bx_n + c + a(x_n + x_{n-1})(x - x_n) + b(x - x_n) + a(x - x_n)^2]\\&\quad - \frac{1}{2}a(x_1 - x_0)(x - x_0) - \frac{1}{2}a(x_n - x_{n-1})(x_n - x)\\&= ax^2 + bx + c,\end{aligned}$$

so we have proven $\mathcal{L}_d(x)$ satisfies the quadric polynomial reproduction property and complete the proof. □

Now, we study the shape-preserving property of $\mathcal{L}_d(x)$. First of all, we calculate the derivatives of $\phi(x) = \frac{1}{3}[x^2 + c^2]^{\frac{3}{2}}$,

$$\phi'(x) = \frac{x(c^2 + x^2)^2}{\sqrt{(c^2 + x^2)^3}}, \qquad \phi''(x) = \frac{(c^2 + x^2)(c^2 + 2x^2)}{\sqrt{(c^2 + x^2)^3}},$$
$$\phi^{(3)}(x) = \frac{3c^2x + 2x^3}{\sqrt{(c^2 + x^2)^3}}, \qquad \phi^{(4)}(x) = \frac{3c^4}{(c^2 + x^2)\sqrt{(c^2 + x^2)^3}}.$$

Theorem 4. *If the data sequence $\{f(x_j)\}_{j=0}^n$ stems from a convex function $f(x) \in C[x_0, x_n]$, then the quasi-interpolation operator $\mathcal{L}_d(x)$ is also a convex function.*

Proof. The second order derivative of $\mathcal{L}_d(x)$ can be calculated as

$$\begin{aligned}\mathcal{L}_d''(x) &= \frac{1}{2}\sum_{j=1}^{n-2}(f[x_{j+2}, x_{j+1}, x_j] - f[x_{j+1}, x_j, x_{j-1}])\omega_j''(x) + f[x_2, x_1, x_0] + f[x_n, x_{n-1}, x_{n-2}]\\&= \frac{1}{2}\sum_{j=1}^{n-3} f[x_{j+2}, x_{j+1}, x_j](\omega_j''(x) - \omega_{j+1}''(x)) + \frac{1}{2}f[x_2, x_1, x_0](2 - \omega_1''(x)) + \frac{1}{2}f[x_n, x_{n-1}, x_{n-2}](2 + \omega_{n-2}''(x)),\end{aligned} \tag{7}$$

because $\phi^{(4)}(x) > 0$ for any $x \in \mathbb{R}$, $\omega_j^{(3)}(x) = \frac{\phi^{(3)}(x-x_j)-\phi^{(3)}(x-x_{j+1})}{x_{j+1}-x_j} = \phi^{(4)}(\varepsilon) > 0$, then $\omega_j''(x)$ is a monotonic function, and $\lim_{x\to\pm\infty}\omega''(x) = \lim_{\varepsilon\to\pm\infty}\phi^{(3)}(\varepsilon) = \pm 2$,

$$-2 < \omega_k''(x) < \omega_m''(x) < 2, \quad k > m. \tag{8}$$

If the data sequence $\{f(x_j)\}_{j=0}^n$ stems from a convex function, then $f[x_{j+2}, x_{j+1}, x_j] > 0$. Combining (7) with (8), we can prove $\mathcal{L}_d''(x) > 0$. Therefore $\mathcal{L}_d(x)$ is strict convexity-preserving, and the proof has been completed. □

Theorem 5. *The quasi-interpolation operator $\mathcal{L}_d(x)$ is strict shape-preserving of order* 3.

Proof. The third order derivative of $\mathcal{L}_d(x)$ is

$$\mathcal{L}_d^{(3)}(x) = \frac{1}{2}\sum_{j=1}^{n-2}(f[x_{j+2}, x_{j+1}, x_j] - f[x_{j+1}, x_j, x_{j-1}])\omega_j^{(3)}(x),$$

If the data sequence $\{f(x_j)\}_{j=0}^n$ stems from a function $f(x)$, and $f'''(x) > 0$, then

$$\begin{aligned}f[x_{j+2}, x_{j+1}, x_j] - f[x_{j+1}, x_j, x_{j-1}] &= f[x_{j+2}, x_{j+1}, x_j, x_{j-1}](x_{j+2} - x_{j-1}),\\&= \frac{f^{(3)}(\xi)}{6}(x_{j+2} - x_{j-1}) \quad (\xi \in (x_{j-1}, x_{j+2}))\\&> 0,\end{aligned}$$

and $\omega_j^{(3)}(x) > 0$, so $\mathcal{L}_d^{(3)}(x) > 0$.

In conclusion, we have proven the quasi-interpolation operator $\mathcal{L}_d(x)$ possesses the shape-preserving of order 3 property and complete the proof. □

3.2. The approximation order of the quasi-interpolation operator $\mathcal{L}_d(x)$

We use the convergence analysis method, which is introduced in [4], to study the approximation capacity of the quasi-interpolation operator $\mathcal{L}_d(x)$.

Theorem 6. *If the second order derivative of $f(x)$ is Lipschitz continuous, then the approximation capacity of $\mathcal{L}_d(x)$ satisfies*

$$\|f - \mathcal{L}_d(x)\|_\infty \le O(h^3) + O(ch^2) + O(c^2h) + O(c^2).$$

Proof. For any fixed $x \in [a, b]$, let $p(y)$ be the local Taylor polynomial of $f(y)$ at the fixed point x, i.e.

$$p(y) = f(x) + f'(x)(y - x) + \frac{f''(x)}{2}(y - x)^2.$$

Meantime, it is obvious that $p[x_{j+2}, x_{j+1}, x_j] = \frac{f''(x)}{2}$, where $p[x_{j+2}, x_{j+1}, x_j]$ denotes the divided difference of $p(y)$. According to Theorem 3, we can get

$$\sum_{j=0}^{n}(x - x_j)^2\alpha_j(x) = 0, \qquad \sum_{j=0}^{n}(x - x_j)\alpha_j(x) = 0, \qquad \sum_{j=0}^{n}\alpha_j(x) = 1,$$

and

$$\begin{aligned}\sum_{j=0}^{n} p(x_j)\alpha_j(x) &= \sum_{j=0}^{n}\left[f(x) + f'(x)(x_j - x) + \frac{f''(x)}{2}(x_j - x)^2\right]\alpha_j(x)\\ &= f(x)\sum_{j=0}^{n}\alpha_j(x) + f'(x)\sum_{j=0}^{n}(x_j - x)\alpha_j(x) + \frac{f''(x)}{2}\sum_{j=0}^{n}(x_j - x)^2\alpha_j(x)\\ &= f(x).\end{aligned}$$

Because $f''(x)$ is Lipschitz continuous, then for any $x_1, x_2 \in [a, b]$, $|f''(x_1) - f''(x_2)| \le C_0|x_1 - x_2|$, where $C_0 = \operatorname{ess\,sup}_{a\le x\le b} |f'''(x)|$. In addition, we can also get $|f(y) - p(y)| \le C_0|y - x|^3$.

$$\begin{aligned}|\mathcal{L}_d(x) - f(x)| &= \left|\sum_{j=0}^{n}(f(x_j) - p(x_j))\alpha_j(x)\right|\\ &\le \frac{1}{2}\left|\sum_{j=1}^{n-2}((f[x_{j+2}, x_{j+1}, x_j] - f[x_{j+1}, x_j, x_{j-1}])\right.\\ &\qquad \left. - (p[x_{j+2}, x_{j+1}, x_j] - p[x_{j+1}, x_j, x_{j-1}]))\omega_j(x)\right| + C_1(x - x_0)^3 + C_2(x_n - x)^3\\ &\le \frac{1}{4}\sum_{j=1}^{n-2}|f''(\xi) - f''(\eta)||\omega_j(x)| + C_1(x - x_0)^3 + C_2(x_n - x)^3 \quad (\xi \in (x_j, x_{j+2}), \eta \in (x_{j-1}, x_{j+1}))\\ &\le \sum_{j=1}^{n-2}\frac{3}{4}C_0h|\omega_j(x)| + C_1(x - x_0)^3 + C_2(x_n - x)^3\\ &\le \frac{3}{4}C_0h\sum_{|x-x_j|\le h}|\omega_j(x)| + \frac{3}{4}C_0h\sum_{|x-x_j|\ge h}|\omega_j(x)| + C_1((x - x_0)^3 + C_2(x_n - x)^3\\ &\le \frac{3}{4}C_0h\sum_{|x-x_j|\le h}|x - x_j|\sqrt{(x - x_j)^2 + c^2} + \frac{3}{4}C_0\int_{|x-t|>h}|x - t|\sqrt{(x - t)^2 + c^2}\mathrm{d}t\\ &\qquad + C_1(x - x_0)^3 + C_2(x_n - x)^3\\ &\le C_0(h + c)^3 + C_1((x - x_0)^3 - \sqrt{[(x - x_0)^2 + c^2]^3}) + C_2((x_n - x)^3 - \sqrt{[(x_n - x)^2 + c^2]^3}),\end{aligned}$$

According to the following two inequalities:

$$\sqrt{[(x - x_0)^2 + c^2]^3} - (x - x_0)^3 \le \frac{3c^2(x - x_0)}{2} + \frac{3c^4}{2(x - x_0)} + \frac{c^6}{2(x - x_0)^3}, \quad c > 0, x - x_0 > h;$$

$$\sqrt{[(x_n - x)^2 + c^2]^3} - (x_n - x)^3 \le \frac{3c^2(x_n - x)}{2} + \frac{3c^4}{2(x_n - x)} + \frac{c^6}{2(x_n - x)^3}, \quad c > 0, x_n - x > h.$$

Table 1
We sample 10 points in [0,1], as $h = 0.1$, $c = 0.01, 0.02, 0.05, 0.1, 0.2$, we observe the approximation capacity of $\mathcal{L}_d(x)$ and $\mathcal{L}_2(x)$

c	0.01	0.02	0.05	0.1	0.2
h	0.1	0.1	0.1	0.1	0.1
$\\|\mathcal{L}_2(x) - f(x)\\|_\infty$	2.90×10^{-3}	6.40×10^{-3}	2.02×10^{-2}	5.36×10^{-2}	1.44×10^{-1}
$\\|\mathcal{L}_d(x) - f(x)\\|_\infty$	2.01×10^{-3}	2.20×10^{-3}	3.40×10^{-3}	7.40×10^{-3}	2.32×10^{-2}

Table 2
We sample 100 points in [0,1], as $h = 0.01$, $c = 0.001, 0.002, 0.005, 0.01, 0.02$, we observe the approximation capacity of $\mathcal{L}_d(x)$ and $\mathcal{L}_2(x)$

c	0.001	0.002	0.005	0.01	0.02
h	0.01	0.01	0.01	0.01	0.01
$\\|\mathcal{L}_2(x) - f(x)\\|_\infty$	8.74×10^{-5}	1.43×10^{-4}	4.17×10^{-4}	1.2×10^{-3}	3.8×10^{-3}
$\\|\mathcal{L}_d(x) - f(x)\\|_\infty$	4.97×10^{-5}	5.19×10^{-5}	6.74×10^{-5}	1.22×10^{-4}	3.41×10^{-4}

Table 3
We sample 1000 points in [0,1], as $h = 0.001$, $c = 0.0001, 0.0002, 0.0005, 0.001, 0.002$, we observe the approximation capacity of $\mathcal{L}_d(x)$ and $\mathcal{L}_2(x)$

c	0.0001	0.0002	0.0005	0.001	0.002
h	0.001	0.001	0.001	0.001	0.001
$\\|\mathcal{L}_2(x) - f(x)\\|_\infty$	5.15×10^{-7}	7.75×10^{-7}	4.86×10^{-6}	1.75×10^{-5}	6.23×10^{-5}
$\\|\mathcal{L}_d(x) - f(x)\\|_\infty$	5.07×10^{-7}	5.29×10^{-7}	6.87×10^{-7}	1.25×10^{-6}	3.49×10^{-6}

Table 4
Let $c = 0.01$, as $h = 0.1, 0.05, 0.025, 0.01, 0.005$, we observe the convergence order of $\mathcal{L}_d(x)$

c	0.01	0.01	0.01	0.01	0.01
h	0.1	0.05	0.025	0.01	0.005
$\\|\mathcal{L}_d(x) - f(x)\\|_\infty$	2.01×10^{-3}	1.20×10^{-3}	3.68×10^{-4}	1.22×10^{-4}	8.64×10^{-5}
r_h	3.165	2.843	2.629	2.346	2.104

then we can get

$$|\mathcal{L}_d(x) - f(x)| \leq C_0h^3 + C_0ch^2 + C_0c^2h + C_1c^2 + C_2c^2,$$

where C_i, $i = 1, 2$ are positive constants and independent of x and h. Now, we have got the approximation order of the quasi-interpolation operator $\mathcal{L}_d(x)$. Obviously, the convergence order of quasi-interpolation $\mathcal{L}_d(x)$ can reach up to 3 provided $c^2 = O(h^3)$ and complete the proof. □

4. Numerical examples

In this section, suppose $f(x) = x^3$ is an approximated function, then we choose different shape parameter c and h to compare the approximation capacity of our quasi-interpolation operator $\mathcal{L}_d(x)$ with that of Wu and Schaback's quasi-interpolation scheme $\mathcal{L}_2(x)$ defined as (2). The comparison results are shown in Tables 1–3. In Tables 1, 2 and 3, we set $h = 0.1, 0.01, 0.001$ respectively, and $c = 0.1h, 0.2h, 0.5h, h, 2h$, then we compute the $\|\mathcal{L}_2(x) - f(x)\|_\infty$ and $\|\mathcal{L}_d(x) - f(x)\|_\infty$. In Table 4, we set $c = 0.01$, $h = 0.1, 0.05, 0.025, 0.01, 0.005$, to observe the convergence rate r_h of $\mathcal{L}_d(x)$ with the variation of h. For sake of simplification, we suppose the sampled points $\{x_i\}_{i=0}^n$ are uniformly distributed.

By analyzing the datas in Tables 1–3, we find that the approximation capacity of a quasi-interpolation scheme is dependent on the shape parameter c and h. Furthermore, the convergence capacity of the quasi-interpolation operator $\mathcal{L}_d(x)$ is better than that of $\mathcal{L}_2(x)$ for the same c and h.

From Table 4, we find that the convergence rate of $\mathcal{L}_d(x)$ can reach up to 3 as $c = O(h^2)$. By these numerical experiments, we can say that our quasi-interpolation scheme $\mathcal{L}_d(x)$ is a very attractive alternative in terms of accuracy.

5. Conclusion

In this paper, we construct a quasi-interpolation operator $\mathcal{L}_d(x)$ to scattered data by the shifts of cubic multiquadric function $\phi_j(x) = \sqrt{[(x - x_j)^2 + c^2]^3}$, which does not require the derivatives of $f(x)$ at endpoints, so it is very practical. Meantime, we have also proven that the operator $\mathcal{L}_d(x)$ possesses good shape-preserving property, the quadric polynomial reproduction property and good convergence capacity, so that it is convenient for people in various application. It is a pity that we only study how to construct quasi-interpolation schemes on one-dimensional space in this paper. In addition, [1] gave a result that the approximation capacity of a quasi-interpolation operator had close relations with the polynomial reproduction property. In work to follow, on the one hand, we plan to construct a quasi-interpolation scheme, which can satisfy much higher-degree polynomial reproduction property. On the other hand, we want to apply this constructing method to multivariate space and construct some quasi-interpolation operators with better properties.

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